PAIR CORRELATIONS AND RANDOM WALKS ON THE INTEGERS

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Abstract. The paper gives conditions for a sequence of fractional parts of real numbers $(\{a_n x\})_{n=1}^{\infty}$ to satisfy a pair correlation estimate. Here x is a fixed non-zero real number and $(a_n)_{n=1}^{\infty}$ is a random walk on the integers.

Let X be a $\mathbb{Z}$-valued function defined on the probability space (Ω, β, P) with characteristic function $\phi(\xi) = \mathbb{E}(e^{iX(\cdot)\xi})$ and let $\chi = \{X_n : n \geq 1\}$ be a sequence of independent copies of X. For a positive integer $n > 0$ let $a_n = X_1 + \cdots + X_n$ and let $a_0 \equiv 0$. The sequence of integers $(a_n)_{n\geq 1}$ is the random walk which we assume to satisfy $|\phi(t) - 1| \leq C|t|$. some $C > 0$. This last property follows for instance if the random walk and its absolute value have finite non-zero mean [Sp p. 62]. In [W1] The condition $|\phi(t) - 1| \geq C|t|$ is said to follow from the assumption that the random is aperiodic and transient – a claim the author was unable to varify. This is then used to deduce a discrepancy estimate for the sequence $(X_n(x))_{n=1}^{\infty}$. This is so for instance (as a consequence of the law of large numbers) if $\mathbb{E}|X| < \infty$ and $\mathbb{E}X \neq 0$ or if X is centred and $\frac{a_n}{n^{\frac{1}{\alpha}}}$ converges in distribution to F_α a stable law of index $\alpha \in (0, 1)$. This second class of examples can be deduced using a local limit theorem of Stone [St]. For a real number x let $X_n(x) = a_n x$. For an interval I let $\chi_I(x)$ denote the characteristic function of the set I. This means that we have $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ otherwise. For a real number y let $\{y\}$ denote its fractional part.

$$V_N(I)(x) = \sum_{1\leq n<m\leq N} \chi_I(\{X_n(x) - X_m(x)\})$$

and then define

$$\Delta_N(x) = \sup_{I\subseteq \mathbb{T}} \left| V_N(I) - \frac{N(N-1)}{2} \mathrm{leb}(I) \right|$$

where the supremum is over all intervals I in the one dimensional torus $\mathbb{T}$. Let $||x|| = \min_{n\in\mathbb{Z}} |x - n| = \min(\{x\}, \{1 - x\})$ let η be a positive real number or infinity. The irrational number x is said to be of type η if η is the supremum of all γ for which $\liminf_{q\to\infty} q^{\gamma}||qx|| = 0$. Using Dirichlet's theorem on diophantine approximation we can deduce for all irrational x that $\liminf_{q\to\infty} q^{\gamma}||qx|| = 0$ so

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$\eta \geq 1$. On the other hand the Thue-Siegel-Roth theorem tells us that for every irrational algebraic x and every $\epsilon > 0$ there exists a constant $c(x, \epsilon) > 0$ such that

$$\left| x - \frac{p}{q} \right| \geq \frac{c(x, \epsilon)}{q^{2+\epsilon}}$$

holds for all coprime integers $q > 0$ and p, so that algebraic η must be of type 1. Lioville numbers can easily be used to show constructively that there exist real numbers that of type strictly greater than 1.

Our theorem is the following.

Theorem. *Suppose $(X_n(x))_{n=1}^{\infty}$ is as described above that x has type $\eta > 1$. Then given $\epsilon > 0$*

$$\Delta_N(x) = o(N^{2-\frac{1}{\eta}+\epsilon}).$$

for P almost all $\omega \in \Omega$.

Let $D_N(x)$ denote the N-term discrepancy of the sequence $(X_n(x))_{n \geq 1}$. See [KN p 88] for the definition. M. Weber [W2 p 411] has given an estimate for almost everywhere behaviour of $D_N(x)$ as N tends to infinity in terms of the type of x and the properties of the the function ϕ. The formulation is however somewhat involved and forgone here. Results like our theorem where $(a_n)_{n=1}^{\infty}$ is fixed and deterministic and x is random are now known. See [NP] for details and further background.

We proceed by a series of lemmas. For real x let $e(x) = e^{2\pi i x}$ and let

$$\theta_N(h) = \sum_{n=1}^{N} e(ha_n x) \qquad (N = 1, 2, \cdots)$$

We need the following lemma taken for [W1].

Lemma 1. *For integers $N \geq R \geq 1$ one has*

$$\mathbb{E}|\theta_N(m) - \theta_R(m)|^2 \leq \min\left(\frac{7(N-R)}{|\phi(mx) - 1|}, N - R \right).$$

Let $(Y_t)_{t=1}^{\infty}$ be a sequence of measurable functions defined on a measure space Ω and then write

$$S_j = \sum_{1 \leq t \leq j} Y_t, \text{ for } j = 1, 2, \cdots.$$

We can define

$$Y_{rs} = \sum_{r \leq t < s} Y_t \quad (= S_s - S_r), \text{ for } r < s,$$

and let $M_n = \sup_{1 \leq j \leq n} |S_j|$. We have the following elementary lemma proved in [NP].

Lemma 2. *For $K \geq 1$,*

$$\int_\Omega M_{2^K}^2(\omega)d\omega \leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\int_\Omega |Y_{\nu 2^{(K+1)-i},(\nu+1)2^{(K+1)-i}}|^2(\omega)d\omega\right).$$

Thus if $K = 1, 2, \cdots$

$$\mathbb{E}|\max_{1\leq j\leq 2^K}\theta_j(m)|^2 \leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\mathbb{E}|\theta_{\nu 2^{(K+1)-i}}(m)-\theta_{(\nu+1)2^{(K+1)-i}}(m)|^2\right)$$

$$\leq (K+1)\left(\sum_{i=1}^{K+1}\sum_{\nu=1}^{2^i-1}\left(\frac{7.2^{(K+1)-i}}{|\phi(mx)-1|}\right)\right)$$

$$\leq (K+1)\left(\sum_{i=1}^{K+1}2^{i-1}\left(\frac{7.2^{(K+1)-i}}{|\phi(mx)-1|}\right)\right)$$

$$\leq \frac{7}{2}(K+1)^2(\frac{2^{(K+1)}}{|\phi(mx)-1|}).$$

Thus, using the Erdős -Turan inequality [KN, p 112-4], we can show that for $L \geq 1$, there exists $C > 0$

$$\mathbb{E}|\max_{1\leq j\leq 2^K}\Delta_j(x)| \leq C\left(\frac{2^{2(K+1)}}{L}+\sum_{h=1}^{L}\frac{1}{h}\left(2^{K+1}+\mathbb{E}\max_{1\leq j\leq 2^K}|\theta_j(h)|^2\right)\right)$$

$$\leq C\left(\frac{2^{2(K+1)}}{L}+\sum_{h=1}^{L}\frac{1}{h}\left(2^{K+1}+\frac{7}{2}(K+1)^2\left(\frac{2^{(K+1)}}{|\phi(hx)-1|}\right)\right)\right). \qquad (1)$$

Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing and such that for any $m \in \mathbb{N}$,

$$\sum_{h=1}^{L}\frac{1}{h|\phi(hx)-1|} \leq \Lambda(L).$$

Then the left hand side of (1)

$$\ll \left(\frac{2^{2(K+1))}}{L}+(\log L)2^{(K+1)}+\Lambda(L)(K+1)^2 2^{(K+1)}\right). \qquad (2)$$

Here of course $\ll$ denotes Vinogradov order notation. Recall that there exists $C > 0$ such that $|1-\phi(t)| \geq C|t|$. Since $\phi(hx) = \phi(\{hx\})$,we therefore have

$$\sum_{h=1}^{L}\frac{1}{h|\phi(hx)-1|} = O\left(\sum_{h=1}^{L}\frac{1}{h||hx||}\right)$$

If x is irrational of type $\eta > 1$ [KN p. 123, Lemma 3.3] for any $\epsilon > 0$ then

$$\sum_{h=1}^{L} \frac{1}{h||hx||} = O(L^{\eta-1+\epsilon})$$

In consequence we can choose $\Lambda(L) = L^{\eta-1+\epsilon}$ the right hand side of (2) is

$$\ll \frac{2^{2(K+1)}}{L} + (\log L)2^{(K+1)} + \Lambda(L)(K+1)^2 2^{(K+1)} \tag{3}$$

Choosing L essentially optimally $2^K \approx L\Lambda(L) = L^{\eta+\epsilon}$ the right hand side of (3) is

$$\ll 2^{(K+1)(2-\frac{1}{\eta}+\epsilon)}(K+1)^2 \tag{4}$$

We now complete the proof of our theorem. Given $\epsilon, \epsilon_0 > 0$, we define

$$E_{\epsilon,\epsilon_0} = \{\omega \in \Omega : \Delta_N(\omega, x) > N^{2-\frac{1}{\eta}+\epsilon}(\log N)^{3+\epsilon_0} \text{ for infinitely many } N\}.$$

We now proceed to show the P measure of E_{ϵ,ϵ_0} is zero. If we denote, for each $K \geq 1$,

$$A_K = \left\{\omega \in \Omega : \max_{2^{K-1} \leq m < 2^K} \Delta_m(\omega, x) > 2^{K(2-\frac{1}{\eta}+\epsilon)} K^{3+\epsilon_0}\right\},$$

then one easily sees that $E_{\epsilon,\epsilon_0} \subseteq \bigcap_{r=1}^{\infty} \bigcup_{K=r}^{\infty} A_K$. Using (4) we can bound

$$\begin{aligned} \mathrm{P}(A_K) \leq & \frac{\mathbb{E}\left|\max_{2^{K-1} \leq m < 2^K} \Delta_m(x)\right|}{2^{K(2-\frac{1}{\eta}+\epsilon)} K^{3+\epsilon_0}} \\ \leq & \frac{C 2^{K(2-\frac{1}{\eta}+\epsilon)} K^2}{K^{3+\epsilon_0} 2^{K(2-\frac{1}{\eta}+\epsilon)}} \leq C K^{-(1+\epsilon_0)}, \end{aligned}$$

for sufficiently large $C > 0$. In particular, we can now observe that

$$\sum_{K=1}^{\infty} \mathrm{P}(A_K) \leq \sum_{K=1}^{\infty} K^{-(1+\epsilon_0)} < +\infty.$$

It follows from the Borel-Cantelli lemma that $\mathrm{P}(E_{\epsilon,\epsilon_0}) = 0$. $\square$

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